

Reflectionless Analytic Difference Operators

III. Hilbert Space Aspects

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Received November 15, 2001; Accepted March 12, 2002

Abstract

In the previous two parts of this series of papers, we introduced and studied a large class of analytic difference operators admitting reflectionless eigenfunctions, focusing on algebraic and function-theoretic features in the first part, and on connections with solitons in the second one. In this third part we study our difference operators from a quantum mechanical viewpoint. We show in particular that for an arbitrary difference operator A from a certain subclass, the reflectionless A -eigenfunctions can be used to construct an unbounded self-adjoint reflectionless operator \hat{A} on $L^2(\mathbb{R}, dx)$, whose action on a suitable core coincides with that of A .

1 Introduction

In this paper we study various quantum mechanical features of a large class of analytic difference operators that admit reflectionless eigenfunctions. Our analytic difference operators (from now on $A\Delta$ Os) are given by

$$A \equiv \exp(-i\partial_x) + V_a(x) \exp(i\partial_x) + V_b(x), \quad (1.1)$$

where $V_a(x)$ and $V_b(x)$ are meromorphic functions with asymptotics

$$\lim_{|\operatorname{Re} x| \rightarrow \infty} V_a(x) = 1, \quad \lim_{|\operatorname{Re} x| \rightarrow \infty} V_b(x) = 0. \quad (1.2)$$

The notion ‘reflectionless eigenfunction’ refers to meromorphic functions $\mathcal{W}(x, p)$ satisfying the eigenvalue equation

$$(A\mathcal{W})(x, p) = (e^p + e^{-p}) \mathcal{W}(x, p), \quad p \in \mathbb{C}, \quad (1.3)$$

and having asymptotics

$$\mathcal{W}(x, p) \sim \begin{cases} e^{ixp}, & \operatorname{Re} x \rightarrow \infty, \\ a(p)e^{ixp}, & \operatorname{Re} x \rightarrow -\infty. \end{cases} \quad (1.4)$$

We make extensive use of previous results in this series of papers, denoting Refs. [1] and [2] by Part I and Part II, resp. In Part I we presented and studied a huge class

of reflectionless $A\Delta O$ s, but here we are concerned with a far smaller class. Indeed, our main interest in this paper is in associating with the $A\Delta O$ A a well-defined self-adjoint operator \hat{A} on the Hilbert space \mathcal{H}_x , where we use the notation

$$\mathcal{H}_y \equiv L^2(\mathbb{R}, dy), \quad (1.5)$$

and we are only able to do so by imposing drastic restrictions on the spectral data in terms of which the coefficient functions ('potentials') V_a and V_b are defined.

It should be mentioned at the outset that we are dealing with exotic territory. In Ref. [3] we studied in great detail some quite special reflectionless $A\Delta O$ s arising in the context of (reduced, 2-particle) relativistic Calogero-Moser systems. (See our contribution Ref. [4] to the NEEDS 2000 Proceedings for the unitary similarity transformation connecting Ref. [3] to the present framework.) From the findings reported in Ref. [3] it is already clear that a complete Hilbert space theory for reflectionless $A\Delta O$ s is not going to amount to a straightforward extension of the well-known results for reflectionless self-adjoint Schrödinger and Jacobi operators. In Sections 3 and 4 of Part II we have summarized the latter results, and we have delineated restrictions on the spectral data for our $A\Delta O$ s such that their reflectionless eigenfunctions can be tied in with the Schrödinger and Jacobi counterparts.

Roughly speaking, we impose similar restrictions in the present paper. We shall be quite precise in Section 2, but in this introduction we try and outline our results with a minimum of technical detail, as this might obscure the basically simple plan of this paper. Before sketching the latter, we add some general remarks yielding more context. To begin with, since we aim to associate with A a self-adjoint operator \hat{A} on \mathcal{H}_x , it is natural to restrict V_a and V_b such that A is at least *formally* self-adjoint. Thus $V_b(x)$ should be real-valued for real x , and $V_a(x) \exp(i\partial_x)$ should be equal to its formal adjoint,

$$[V_a(x) \exp(i\partial_x)]^* = \exp(i\partial_x) \overline{V_a(x)} = \overline{V_a(x-i)} \exp(i\partial_x), \quad x \in \mathbb{R}. \quad (1.6)$$

Hence we need

$$V_b^*(x) = V_b(x), \quad V_a^*(x) = V_a(x-i), \quad x \in \mathbb{C}, \quad (1.7)$$

where the $*$ denotes the conjugate meromorphic function,

$$f^*(x) \equiv \overline{f(\bar{x})}, \quad x \in \mathbb{C}. \quad (1.8)$$

From now on we restrict attention to potentials satisfying (1.7). Then a natural strategy would be to try and find a dense subspace \mathcal{C} in \mathcal{H}_x on which A is well defined and symmetric. Thus, \mathcal{C} should consist of square-integrable functions $f(x)$, $x \in \mathbb{R}$, that are restrictions to \mathbb{R} of functions that have suitable analyticity properties for $|\operatorname{Im} x| \leq 1$, so that there is an unambiguous meaning for $f(x \pm i)$; then the function

$$(Af)(x) \equiv f(x-i) + V_a(x)f(x+i) + V_b(x)f(x), \quad x \in \mathbb{R}, \quad (1.9)$$

should be square-integrable, and one should have

$$(f, Ag) = (Af, g), \quad f, g \in \mathcal{C}. \quad (1.10)$$

Assuming such a dense subspace has been isolated, one can try and study the existence and uniqueness of self-adjoint extensions. Indeed, the symmetric operator A on \mathcal{C} is unbounded (due to the shifts), so it might not have any self-adjoint extensions or a (finite- or infinite-dimensional) family of self-adjoint extensions.

In any event, assuming some self-adjoint operator \hat{A} has been associated with A via this procedure, one can define its being ‘reflectionless’ solely in terms of time-dependent Hilbert space scattering theory, as follows.

First of all, there is a natural ‘free’ dynamics $\exp(-it\hat{A}_0)$ with which the ‘interacting’ dynamics $\exp(-it\hat{A})$ can be compared. Indeed, the AΔO

$$A_0 \equiv \exp(-i\partial_x) + \exp(i\partial_x) \quad (1.11)$$

gives rise to an obvious self-adjoint operator \hat{A}_0 on \mathcal{H}_x , namely, the transform

$$\hat{A}_0 \equiv \mathcal{F}_0 M \mathcal{F}_0^{-1} \quad (1.12)$$

of the self-adjoint multiplication operator

$$(Mf)(p) \equiv 2 \cosh(p) f(p), \quad f \in \mathcal{D}(M), \quad (1.13)$$

with maximal domain $\mathcal{D}(M)$ under Fourier transformation

$$\mathcal{F}_0 : \mathcal{H}_p \rightarrow \mathcal{H}_x, \quad f(p) \mapsto (2\pi)^{-1/2} \int_{-\infty}^{\infty} dp e^{ixp} f(p). \quad (1.14)$$

(Recall our notation (1.5).)

Now assume that the (strong) limits of the operator family $\exp(it\hat{A})\exp(-it\hat{A}_0)$ for $t \rightarrow \pm\infty$ exist and have equal range. Denoting these isometric wave operators by W_{\pm} , the corresponding S -operator

$$S_x \equiv W_+^* W_- \quad (1.15)$$

is unitary. Since it commutes with the free evolution $\exp(-it\hat{A}_0)$, its transform

$$S_p \equiv \mathcal{F}_0^{-1} S_x \mathcal{F}_0 \quad (1.16)$$

to \mathcal{H}_p is of the form

$$(S_p f)(p) = T(p) f(p) + R(p) f(-p), \quad f \in \mathcal{H}_p, \quad (1.17)$$

for certain functions $T(p)$, $R(p)$. Then the dynamics \hat{A} is by definition reflectionless when $R(p)$ vanishes identically.

Our summary of these notions from time-dependent scattering theory (about which a wealth of pertinent information can be found in Ref. [5]) serves a twofold purpose. First, it has enabled us to sketch a *general* scenario in which the concept of ‘reflectionless self-adjoint AΔO’ makes sense and can be studied. Second, we actually follow a quite different strategy in this paper, but time-dependent scattering theory does play a crucial role. Thus we are now better prepared to sketch our *special* setting, and compare it to the above approach.

The main difference consists in our definition of the self-adjoint operator \hat{A} : It hinges on using the quite special A -eigenfunctions $\mathcal{W}(x, p)$. (Note that in the general setting just

sketched, eigenfunctions of the A Δ O A need not be and are not mentioned.) Specifically, the eigenfunction transform

$$\mathcal{F} : \mathcal{H}_p \rightarrow \mathcal{H}_x, \quad f(p) \mapsto (2\pi)^{-1/2} \int_{-\infty}^{\infty} dp \mathcal{W}(x, p) f(p) \quad (1.18)$$

plays a decisive role in defining \hat{A} .

We have already seen the simplest example of this approach. Indeed, for the A Δ O A_0 (1.11) we defined the associated Hilbert space operator \hat{A}_0 by using the A_0 -eigenfunctions $\exp(ixp)$, $p \in \mathbb{R}$, cf. (1.14). The unitarity of \mathcal{F}_0 is crucial here: Invertibility of \mathcal{F}_0 would not be enough for (1.12) to give rise to a self-adjoint operator \hat{A}_0 . To compare with the general strategy, we mention that a domain \mathcal{C}_0 of essential self-adjointness as considered above is for instance given by $\mathcal{F}_0(C_0^\infty(\mathbb{R}))$. The point is, however, that the latter domain cannot be readily described in terms of the position space \mathcal{H}_x . Moreover, even for A_0 there exists an infinite-dimensional family of distinct domains of essential self-adjointness yielding distinct reflectionless self-adjoint operators on \mathcal{H}_x . (This can already be concluded from the special cases studied in Ref. [3], cf. also Ref. [4]. The present more general case yields a much larger family, as shown at the end of Section 4.)

To appreciate the latter state of affairs, and, accordingly, the *choice* involved in taking (1.18) as a starting point, a crucial feature of A -eigenfunctions should be recalled: They remain eigenfunctions with the same eigenvalue after they are multiplied by an arbitrary meromorphic function with period i . In particular, this entails that when an A Δ O of the form (1.1)–(1.2) admits a reflectionless eigenfunction $\mathcal{W}(x, p)$ satisfying (1.3)–(1.4), it also admits a reflectionless eigenfunction $\tilde{\mathcal{W}}(x, p)$ with any other function $\tilde{a}(p)$ in its asymptotics (1.4). Indeed, we need only set

$$\tilde{\mathcal{W}}(x, p) \equiv \mu(x, p) \mathcal{W}(x, p), \quad (1.19)$$

with

$$\mu(x, p) \equiv (e^{2\pi x} + \tilde{a}(p)a(p)^{-1}e^{-2\pi x}) / (e^{2\pi x} + e^{-2\pi x}), \quad (1.20)$$

to obtain a new eigenfunction with these features.

Now there is no reason to expect that when the operator (1.18) is unitary (or at least isometric) for a particular choice of $\mathcal{W}(x, p)$, it is still unitary/isometric for eigenfunctions $\tilde{\mathcal{W}}(x, p)$ as just described. Indeed, in the case of \mathcal{F}_0 it can be proved that multipliers of the form (1.20) destroy unitarity. But as already alluded to, in this case there does exist an infinite-dimensional family of i -periodic multipliers for which unitarity is preserved.

In our approach, then, the Hilbert space features of the eigenfunction transform \mathcal{F} (1.18) are of primary importance. We are able to establish the relevant functional-analytic features by using the considerable amount of explicit algebraic and function-theoretic information gathered in Parts I and II. In particular, the surprising connection to classical N -particle relativistic Calogero–Moser systems established in II Section 5 is instrumental in obtaining important additional information of the same character, whose derivation we have relegated to Appendix A.

In outline, we solve the pertinent Hilbert space problems as follows. First of all, we choose the spectral data in terms of which $\mathcal{W}(x, p)$ is defined such that the transform \mathcal{F} (1.18) is a bounded operator on \mathcal{H}_p . This is already the case whenever $\mathcal{W}(x, p)$ has no poles for real x , which is a weak restriction. This choice also ensures that no

nontrivial $C_0^\infty(\mathbb{R})$ -function is annihilated by \mathcal{F} , cf. Lemma 2.1. As a consequence, we are entitled to define an operator \hat{A} on the subspace

$$\mathcal{P} \equiv \mathcal{F}C_0^\infty(\mathbb{R}), \quad (1.21)$$

by setting

$$\hat{A}\mathcal{F}f \equiv \mathcal{F}Mf, \quad f \in C_0^\infty(\mathbb{R}), \quad (1.22)$$

where M is defined by (1.13). (The relation to the AΔO A is also clarified in Lemma 2.1.)

A far more drastic restriction on the spectral data now ensures that $\mathcal{W}(x, p)$ has no poles for $\text{Im } x \in [-1, 0]$. Note that this striking feature is generically destroyed when $\mathcal{W}(x, p)$ is multiplied by i -periodic multipliers $\mu(x, p)$ with constant limits for $|\text{Re } x| \rightarrow \infty$. (Indeed, by Liouville's theorem the latter must have poles in a period strip to be nonconstant. On the other hand, these poles might occur at the same locations as zeros (counting multiplicities) of $\mathcal{W}(x, p)$ in the strip $\text{Im } x \in [-1, 0]$, in which case $\mu(x, p)\mathcal{W}(x, p)$ would still be pole-free in this strip.)

Due to the absence of these critical poles, we are able to show that the operator \hat{A} is *symmetric* on \mathcal{P} . This involves considerable work, whereas the next step is quite easy: An application of Nelson's analytic vector theorem [6] yields essential self-adjointness of \hat{A} on \mathcal{P} , cf. Lemma 2.2. Denoting the self-adjoint extension by the same symbol, we obtain a unitary one-parameter group $\exp(-it\hat{A})$ on the closure $\overline{\mathcal{P}}$ of the subspace \mathcal{P} . In general, this is a proper subspace of \mathcal{H}_x (that is, in general \mathcal{F} is not onto \mathcal{H}_x), and we now extend \hat{A} *provisionally* to a self-adjoint operator acting in \mathcal{H}_x by putting it equal to an arbitrarily chosen self-adjoint operator on the orthogonal complement \mathcal{P}^\perp . (At this stage we do not yet know that the latter space is spanned by eigenfunctions of the AΔO A , so this provisional extension cannot be avoided.)

Our next goal consists in handling the time-dependent scattering theory of the interacting dynamics $\exp(-it\hat{A})$, as compared to the free dynamics $\exp(-it\hat{A}_0)$. We do this in Section 3, the most important result being that the wave operators can be written in terms of \mathcal{F} , cf. Theorem 3.2. From our explicit formulas it is then clear by inspection that \mathcal{F} is an isometry. Moreover, they show that the S -matrix S_p (1.16) is the one expected from time-independent scattering theory. (That is, the S -matrix expected from the asymptotics (1.4) of the eigenfunction.)

In Section 4 we complete our analysis by clarifying the state of affairs on \mathcal{P}^\perp : This space is spanned by finitely many pairwise orthogonal eigenfunctions $\mathcal{W}(x, r_n)$, $r_n \in i(0, \pi)$, $n = 1, \dots, N_+$, with distinct real eigenvalues $2 \cosh(r_n)$. Thus the definition of \hat{A} can be completed by requiring that its action on \mathcal{P}^\perp be equal to that of A , just as its action on \mathcal{P} . The key to understanding \mathcal{P}^\perp is an explicit formula for $\mathcal{F}\mathcal{F}^*$, which we obtain along the same lines as similar formulas for the special cases we studied in Ref. [3]. (Since we have no duality properties available in the present general framework, we cannot proceed in this way to obtain the isometry formula $\mathcal{F}^*\mathcal{F} = \mathbf{1}$. Instead, we exploit the isometry of wave operators, cf. Section 3.)

We conclude Section 4 with an appraisal of some special cases. Of particular interest is the subclass for which no point spectrum occurs in the spectral resolution of \hat{A} (corresponding to $N_+ = 0$). This infinite-dimensional family has no analog for reflectionless self-adjoint Schrödinger and Jacobi operators. We use it to illustrate the ambiguity issue discussed above.

2 Essential self-adjointness on the domain \mathcal{P}

We begin by recalling how our class of A Δ O's A (1.1) and the associated reflectionless eigenfunctions $\mathcal{W}(x, p)$ are obtained from ‘spectral data’ $(r, \mu(x))$. The vector $r = (r_1, \dots, r_N)$, with $N \in \mathbb{N}^*$, consists of complex numbers satisfying

$$e^{r_m} \neq e^{\pm r_n}, \quad 1 \leq m < n \leq N, \quad (2.1)$$

and

$$\operatorname{Im} r_n \in \begin{cases} (0, \pi), & n = 1, \dots, N_+, \\ (-\pi, 0), & n = N - N_- + 1, \dots, N, \end{cases} \quad (2.2)$$

where

$$N_+, N_- \in \{0, \dots, N\}, \quad N_+ + N_- = N. \quad (2.3)$$

The vector $\mu(x) = (\mu_1(x), \dots, \mu_N(x))$ consists of meromorphic functions satisfying

$$\mu_n(x+i) = \mu_n(x), \quad \lim_{|\operatorname{Re} x| \rightarrow \infty} \mu_n(x) = c_n, \quad c_n \in \mathbb{C}^*, \quad n = 1, \dots, N. \quad (2.4)$$

These ‘minimal’ restrictions on (r, μ) are in force throughout this paper. (When the need arises, we specify additional restrictions.)

Now we define a Cauchy matrix

$$C_{mn} \equiv \frac{1}{e^{r_m} - e^{-r_n}}, \quad m, n = 1, \dots, N, \quad (2.5)$$

and a diagonal matrix

$$D(x) \equiv \operatorname{diag}(d(r_1, \mu_1; x), \dots, d(r_N, \mu_N; x)), \quad (2.6)$$

where

$$d(\rho, \nu; x) \equiv \begin{cases} \nu(x)e^{-2i\rho x}, & \operatorname{Im} \rho \in (0, \pi), \\ \nu(x)e^{-2i(\rho+i\pi)x}, & \operatorname{Im} \rho \in (-\pi, 0). \end{cases} \quad (2.7)$$

Then the potentials V_a , V_b and wave function \mathcal{W} are defined via the solution to the system

$$(D(x) + C)R(x) = \zeta, \quad \zeta \equiv (1, \dots, 1)^t, \quad (2.8)$$

by

$$V_a(x) \equiv \left(1 + \sum_{n=1}^N e^{r_n} R_n(x)\right) \left(1 + \sum_{n=1}^N e^{r_n} R_n(x+i)\right)^{-1}, \quad (2.9)$$

$$V_b(x) \equiv \sum_{n=1}^N (R_n(x-i) - R_n(x)), \quad (2.10)$$

$$\mathcal{W}(x, p) \equiv e^{ixp} \left(1 - \sum_{n=1}^N \frac{R_n(x)}{e^p - e^{-r_n}}\right). \quad (2.11)$$

Of course, it is far from obvious that these definitions entail the eigenvalue equation (1.3), but this is shown in I Theorem 2.3. In contrast, the asymptotics

$$\lim_{\operatorname{Re} x \rightarrow \infty} R(x) = 0, \quad \lim_{\operatorname{Re} x \rightarrow -\infty} R(x) = C^{-1}\zeta, \quad (2.12)$$

easily follows from (2.5)–(2.8), cf. also I Lemma 2.1. Using (2.12), one obtains the asymptotics (1.2) and (1.4), with

$$a(p) \equiv \prod_{n=1}^N \frac{e^p - e^{r_n}}{e^p - e^{-r_n}}, \quad (2.13)$$

cf. I Theorem 2.3.

We are not able to associate a self-adjoint operator on \mathcal{H}_x to the AΔO A unless we impose further restrictions on the data (r, μ) . But to prove boundedness of the eigenfunction transform \mathcal{F} (1.18) and a few more salient features, we only need a quite weak assumption, as detailed in the next lemma. (Indeed, for generic spectral data satisfying (2.1)–(2.4) the meromorphic function $R(x)$ has no poles on the real axis.)

Lemma 2.1. *Assume that the solution $R(x)$ to (2.8) has no poles for real x . Then the operator \mathcal{F} (1.18) is bounded. For all $\phi \in C_0^\infty(\mathbb{R})$ the function $(\mathcal{F}\phi)(x)$ extends to a meromorphic function satisfying*

$$A(\mathcal{F}\phi)(x) = (\mathcal{F}M\phi)(x), \quad x \in \mathbb{C}, \quad (2.14)$$

where A is the AΔO (1.1) and M the multiplication operator (1.13). Moreover, we have

$$\operatorname{Ker}(\mathcal{F}) \cap C_0^\infty(\mathbb{R}) = \{0\}. \quad (2.15)$$

Proof. In view of the asymptotics (2.12) and absence of real poles, the function $R(x)$ is bounded for real x . Due to the restrictions (2.2), the functions $(e^p - e^{-r_n})^{-1}$, $n = 1, \dots, N$, are bounded for real p . Hence $\mathcal{W}(x, p)$ (2.11) is bounded for real x, p . Choosing $\phi(p) \in C_0^\infty(\mathbb{R})$, we may write

$$(\mathcal{F}\phi)(x) = (\mathcal{F}_0\phi)(x) - \sum_{n=1}^N R_n(x)(\mathcal{F}_0\phi_n)(x), \quad (2.16)$$

with

$$\phi_n(p) \equiv (e^p - e^{-r_n})^{-1} \phi(p), \quad (2.17)$$

cf. (1.14). Since Fourier transformation \mathcal{F}_0 is a bounded operator, and the multiplication operators occurring here are bounded, too, boundedness of \mathcal{F} follows.

To prove the second assertion, we recall the easily verified fact that the Fourier transform of a $C_0^\infty(\mathbb{R})$ -function extends to an entire function. Since we have $\phi, \phi_1, \dots, \phi_N \in C_0^\infty(\mathbb{R})$, and since $R_1(x), \dots, R_N(x)$ are meromorphic, it is clear from (2.16) that $(\mathcal{F}\phi)(x)$ extends to a meromorphic function. The action of A on this function yields the meromor-

phic function

$$\begin{aligned}
& (\mathcal{F}_0\phi)(x-i) - \sum_{n=1}^N R_n(x-i)(\mathcal{F}_0\phi_n)(x-i) \\
& + V_a(x) \left[(\mathcal{F}_0\phi)(x+i) - \sum_{n=1}^N R_n(x+i)(\mathcal{F}_0\phi_n)(x+i) \right] \\
& + V_b(x) \left[(\mathcal{F}_0\phi)(x) - \sum_{n=1}^N R_n(x)(\mathcal{F}_0\phi)(x) \right]. \tag{2.18}
\end{aligned}$$

For all x for which the functions $R(x)$, $R(x \pm i)$, $V_a(x)$ and $V_b(x)$ have no poles, this can be rewritten as the absolutely convergent integral

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} dp [\mathcal{W}(x-i, p) + V_a(x)\mathcal{W}(x+i, p) + V_b(x)\mathcal{W}(x, p)] \phi(p). \tag{2.19}$$

Thanks to the eigenvalue equation (1.3), the function in square brackets amounts to $2 \cosh(p)\mathcal{W}(x, p)$, yielding (2.14).

To prove (2.15), we assume $\phi \in C_0^\infty(\mathbb{R})$ satisfies $\mathcal{F}\phi = 0$. By (2.16) we then have

$$(\mathcal{F}_0\phi)(x) = \sum_{n=1}^N R_n(x)(\mathcal{F}_0\phi_n)(x). \tag{2.20}$$

Consider now the function $e^{ax}R_n(x)$, $x \in \mathbb{R}$, $a \geq 0$. Due to (2.12), it is bounded for $x \rightarrow -\infty$. The $\operatorname{Re} x \rightarrow \infty$ asymptotics of $R(x)$ can be sharpened to an exponential decay, so that $e^{ax}R(x)$ is also bounded at ∞ , provided $a \in [0, c]$, with c small enough. (The pertinent asymptotic decay easily follows from (2.8), cf. I(2.41)–(2.42).) Now $(\mathcal{F}_0\phi_n)(x)$ is a Schwartz space function, so it readily follows that the functions

$$f_n(z) \equiv (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx e^{-ixz} R_n(x)(\mathcal{F}_0\phi_n)(x), \quad n = 1, \dots, N, \tag{2.21}$$

are well defined for $\operatorname{Im} z \in [0, c]$ and analytic for $\operatorname{Im} z \in (0, c)$. Moreover, a dominated convergence argument yields

$$\lim_{a \downarrow 0} f_n(p+ia) = f_n(p), \quad n = 1, \dots, N, \tag{2.22}$$

uniformly for real p .

From (2.20) we deduce that the function

$$f(z) \equiv (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx e^{-ixz} (\mathcal{F}_0\phi)(x) = \sum_{n=1}^N f_n(z) \tag{2.23}$$

is also well defined for $\operatorname{Im} z \in [0, c]$ and analytic for $\operatorname{Im} z \in (0, c)$. Since it satisfies

$$\lim_{a \downarrow 0} f(p+ia) = \phi(p), \tag{2.24}$$

uniformly for $p \in \mathbb{R}$, and $\phi(p) \in C_0^\infty(\mathbb{R})$ vanishes on an open set, Painlevé's lemma entails $\phi = 0$. Hence (2.15) follows. ■

In view of (2.15), any vector in the subspace \mathcal{P} (1.21) can be written as $\mathcal{F}\phi$, $\phi \in C_0^\infty(\mathbb{R})$, in a unique way. Therefore, the operator \hat{A} (1.22) is a well-defined linear operator on \mathcal{P} , whose action coincides with that of the AΔO A , cf. (2.14). Obviously, \hat{A} leaves \mathcal{P} invariant. Moreover, we clearly have

$$\|\hat{A}^n \mathcal{F}\phi\| \leq c^n \|\mathcal{F}\| \|\phi\|, \quad n \in \mathbb{N}, \quad (2.25)$$

with $c > 0$ depending on $\text{supp}(\phi)$. Thus \mathcal{P} consists of analytic vectors for \hat{A} . In view of Nelson's analytic vector theorem [6] it now suffices for essential self-adjointness of \hat{A} on \mathcal{P} that \hat{A} is symmetric on \mathcal{P} .

Next, we detail assumptions on the spectral data that suffice to prove this critical symmetry property. We do this in four steps, each of which adds a restriction. This enables us to use the less restrictive intermediate assumptions whenever we can show their sufficiency for the result at hand. (In most cases, however, we do not know to what extent these assumptions are necessary.)

Our first step consists in imposing *formal* self-adjointness (1.7). This property is ensured by requiring that r_1, \dots, r_N be purely imaginary and that the functions $ie^{-r_n} \mu_n(x)$ be real-valued for $n = 1, \dots, N$ and real x , cf. I Theorem D.1. Our second step consists in requiring that $\mu(x)$ be constant. Thus our second assumption comes down to

$$ir_n \in \mathbb{R}, \quad ie^{-r_n} \mu_n(x) \equiv \nu_n^{-1} \in \mathbb{R}^*, \quad n = 1, \dots, N. \quad (2.26)$$

Our third requirement reads

$$\nu_1, \dots, \nu_N \in (0, \infty). \quad (2.27)$$

There are explicit examples available where the second assumption (2.26) and the assumption of Lemma 2.1 are satisfied, but the third assumption (2.27) and symmetry of \hat{A} are violated. These examples can be gleaned from Ref. [3], but we do not spell out the details here.

Our fourth and final restriction can be most easily phrased in terms of the τ -function

$$\tau(x) \equiv |\mathbf{1}_N + CD(x)^{-1}|. \quad (2.28)$$

In view of our second restriction (2.26), $\tau(x)$ is an entire function, cf. (2.6)–(2.7). Likewise, from (2.8) we see that $R(x)$ can be written

$$R(x) = E(x)/\tau(x), \quad (2.29)$$

where $E(x)$ is an entire function. Our fourth requirement is now that $\tau(x)$ have no zeros for $\text{Im } x \in [-1, 0]$. This entails in particular that $R(x)$ has no real poles (the assumption of Lemma 2.1).

In Appendix A we prove that when N_+ or N_- equals N (so that all r_n lie either on the positive or on the negative imaginary axis), the third restriction entails the fourth one, cf. Lemma A.1. For $N_+ N_- > 0$ this is presumably still true for generic spectral data, but we were unable to prove this. Explicit examples we do not present here show that our fourth requirement *is* stronger than the third one. In any event, in Appendix A we also prove that for $N_+ N_- > 0$ the third restriction (2.27) together with the requirement

$$-ir_1, \dots, -ir_{N_+}, -ir_{N_++1} + \pi, \dots, -ir_N + \pi \in (0, \pi/2] \quad (2.30)$$

are sufficient to obtain the fourth one, cf. Lemma A.2.

We continue by showing that our fourth assumption suffices for essential self-adjointness of \hat{A} on its definition domain \mathcal{P} . In the proof we use one property of $\tau(x)$ that cannot be found in Parts I and II, namely,

$$\tau^*(x) = \tau(x - i). \quad (2.31)$$

This formula is an easy consequence of (2.28) and the relations

$$\overline{C} = -C^t, \quad D^*(x) = -D(x - i), \quad (2.32)$$

which follow from (2.26). (Cf. also Appendix A and I Appendix D.)

Lemma 2.2. *Assume that the data satisfy (2.26)–(2.27) and that $\tau(x)$ (2.28) has no zeros for $\text{Im } x \in [-1, 0]$. Then the operator \hat{A} defined by (1.22) is essentially self-adjoint on \mathcal{P} (1.21).*

Proof. As already detailed, it suffices to prove symmetry of \hat{A} on \mathcal{P} . (Recall the paragraph containing (2.25).) For this purpose we fix $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$ and consider $(\mathcal{F}\phi_1, \hat{A}\mathcal{F}\phi_2)$. By definition, this equals

$$\begin{aligned} (\mathcal{F}\phi_1, \mathcal{F}M\phi_2) &= \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-R}^R dx \left(\int_{-\infty}^{\infty} dq \mathcal{W}(x, q) \phi_1(q) \right)^- \\ &\quad \times \left(\int_{-\infty}^{\infty} dp \mathcal{W}(x, p) 2 \cosh(p) \phi_2(p) \right) \\ &= \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} dq \overline{\phi_1(q)} \int_{-\infty}^{\infty} dp \phi_2(p) \int_{-R}^R dx \overline{\mathcal{W}(x, q)} 2 \cosh(p) \mathcal{W}(x, p), \end{aligned} \quad (2.33)$$

where we used Fubini's theorem in the last step. Rewriting $(\hat{A}\mathcal{F}\phi_1, \mathcal{F}\phi_2)$ in the same way, we obtain

$$(\mathcal{F}\phi_1, \hat{A}\mathcal{F}\phi_2) - (\hat{A}\mathcal{F}\phi_1, \mathcal{F}\phi_2) = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} dq \overline{\phi_1(q)} \int_{-\infty}^{\infty} dp \phi_2(p) I_R(p, q), \quad (2.34)$$

where

$$I_R(p, q) \equiv \int_{-R}^R dx \overline{\mathcal{W}(x, q)} \mathcal{W}(x, p) (2 \cosh(p) - 2 \cosh(q)), \quad p, q \in \mathbb{R}. \quad (2.35)$$

Next, we invoke the eigenvalue equation (1.3) and the notation (1.8) to rewrite $I_R(p, q)$ as

$$\begin{aligned} &\int_{-R}^R dx (\mathcal{W}^*(x, q) [\mathcal{W}(x - i, p) + V_a(x) \mathcal{W}(x + i, p) + V_b(x) \mathcal{W}(x, p)] \\ &\quad - [\mathcal{W}^*(x + i, q) + V_a^*(x) \mathcal{W}^*(x - i, q) + V_b^*(x) \mathcal{W}^*(x, q)] \mathcal{W}(x, p)). \end{aligned} \quad (2.36)$$

Recalling (1.7), we obtain

$$I_R(p, q) = \int_{-R}^R dx (J(x, p, q) - J(x + i, p, q)), \quad (2.37)$$

where

$$J(x, p, q) \equiv \mathcal{W}^*(x, q)\mathcal{W}(x - i, p) - V_a(x - i)\mathcal{W}^*(x - i, q)\mathcal{W}(x, p). \quad (2.38)$$

Now from (2.11) and (2.29) we have

$$\mathcal{W}(x, p) = e^{ixp} \left(1 - \sum_{n=1}^N \frac{E_n(x)}{\tau(x)} \frac{1}{e^p - e^{-r_n}} \right), \quad (2.39)$$

with $E_n(x)$ entire. Using (2.31), we deduce

$$\mathcal{W}^*(x, q) = e^{-ixq} \left(1 - \sum_{n=1}^N \frac{E_n^*(x)}{\tau(x - i)} \frac{1}{e^q - e^{r_n}} \right), \quad q \in \mathbb{R}. \quad (2.40)$$

From II(2.34) we also have the identity

$$V_a(x) = \frac{\tau(x + i)\tau(x - i)}{\tau(x)^2}. \quad (2.41)$$

When we substitute (2.39)–(2.41) in $J(x, p, q)$, we can write the result as

$$\begin{aligned} J(x, p, q) &= e^{-ixq} \left(1 - \sum_{n=1}^N \frac{E_n^*(x)}{\tau(x - i)} \frac{1}{e^q - e^{r_n}} \right) e^{i(x-i)p} \left(1 - \sum_{k=1}^N \frac{E_k(x - i)}{\tau(x - i)} \frac{1}{e^p - e^{-r_k}} \right) \\ &\quad - \frac{1}{\tau(x - i)^2} e^{-i(x-i)q} \left(\tau(x - 2i) - \sum_{n=1}^N E_n^*(x - i) \frac{1}{e^q - e^{r_n}} \right) \\ &\quad \times e^{ixp} \left(\tau(x) - \sum_{k=1}^N E_k(x) \frac{1}{e^p - e^{-r_k}} \right). \end{aligned} \quad (2.42)$$

The point of doing so is that this representation shows that $J(x, p, q)$ has no poles on and inside the rectangular contour Γ in the x -plane connecting $-R$, R , $R + i$, $-R + i$. (Indeed, by assumption $\tau(x)$ is zero-free for $\text{Im } x \in [-1, 0]$.)

As a consequence, the contour integral

$$C_R(p, q) \equiv \oint_{\Gamma} dx J(x, p, q) \quad (2.43)$$

vanishes by Cauchy's theorem. On the other hand, $C_R(p, q)$ equals $I_R(p, q)$ plus the integrals over the vertical sides of Γ . Thus we infer

$$\begin{aligned} I_R(p, q) &= \left(\int_R^{R+i} + \int_{-R+i}^{-R} \right) dx \\ &\quad \times [-\mathcal{W}^*(x, q)\mathcal{W}(x - i, p) + V_a(x - i)\mathcal{W}^*(x - i, q)\mathcal{W}(x, p)]. \end{aligned} \quad (2.44)$$

In order to handle the right boundary term for $R \rightarrow \infty$, we substitute

$$\mathcal{W}(x, p) \equiv e^{ixp} + \rho_+(x, p), \quad \mathcal{W}^*(x, q) \equiv e^{-ixq} + \rho_+^*(x, q), \quad (2.45)$$

$$V_a(x) \equiv 1 + \rho(x), \quad (2.46)$$

in the first integral of (2.44). An inspection of (the proofs of) Lemmas 2.1, 2.2 and Theorem 2.3 in Part I now reveals

$$\rho_+(x, p), \rho_+^*(x, q), \rho(x) \rightarrow 0, \quad \operatorname{Re} x \rightarrow \infty, \quad (2.47)$$

uniformly for p, q and $\operatorname{Im} x$ in \mathbb{R} -compacts. An easy dominated convergence argument then shows that the contribution to (2.34) of terms containing at least one ρ vanishes. Thus we are left with

$$\begin{aligned} & \int_R^{R+i} \left(-e^{-ixq+i(x-i)p} + e^{-i(x-i)q+ixp} \right) \\ &= -4ie^{iR(p-q)} \sinh((p+q)/2) \frac{\sinh((p-q)/2)}{p-q}. \end{aligned} \quad (2.48)$$

Substituting this in (2.34), we can transform to sum and difference variables to infer that the contribution of this term vanishes by virtue of the Riemann–Lebesgue lemma.

Next, we substitute

$$\mathcal{W}(x, p) = a(p)e^{ixp} + \rho_-(x, p), \quad \mathcal{W}^*(x, q) = a^*(q)e^{-ixq} + \rho_-^*(x, q), \quad (2.49)$$

and (2.46) in the second integral of (2.44). Using the alternative representation I(2.49) of $a(p)$, we obtain

$$\begin{aligned} \rho_-(x, p) &= \sum_{n=1}^N ((C^{-1}\zeta)_n - R_n(x)) / (e^p - e^{-r_n}) \\ &= \sum_{n=1}^N (C^{-1}D(x)R(x))_n / (e^p - e^{-r_n}), \end{aligned} \quad (2.50)$$

where we used (2.8) in the second step. As before, we now infer

$$\rho_-(x, p), \rho_-^*(x, q), \rho(x) \rightarrow 0, \quad \operatorname{Re} x \rightarrow -\infty, \quad (2.51)$$

uniformly for p, q and $\operatorname{Im} x$ in \mathbb{R} -compacts. Thus it remains to consider the contribution of

$$4ia^*(q)a(p) \exp[-iR(p-q)] \sinh((p+q)/2) \sinh((p-q)/2)/(p-q) \quad (2.52)$$

to (2.34). As before, this vanishes by the Riemann–Lebesgue lemma. Hence the rhs of (2.34) vanishes, entailing symmetry of \hat{A} on \mathcal{P} . \blacksquare

With the assumptions of the lemma in effect, we can take the closure of the operator \hat{A} on \mathcal{P} to obtain a self-adjoint operator. The latter acts on a dense subspace of the Hilbert space

$$\mathcal{H}_x(\mathcal{F}) \equiv \overline{\operatorname{Ran}(\mathcal{F})}. \quad (2.53)$$

It follows from the isometry of \mathcal{F} , which we prove in the next section, that the range of \mathcal{F} is actually closed. At this stage, however, we only know \mathcal{F} is bounded and we have no information about $\operatorname{Ran}(\mathcal{F})$ and its orthogonal complement.

Until further notice, we denote by \hat{A} the self-adjoint operator on \mathcal{H}_x that acts as the closure of \hat{A} on $\mathcal{H}_x(\mathcal{F})$, and as an arbitrarily chosen self-adjoint operator on the orthogonal complement $\mathcal{H}_x(\mathcal{F})^\perp$. The results of the next section are independent of the latter choice.

3 Time-dependent scattering theory

Throughout this section, the assumptions of Lemma 2.2 are in effect. With the self-adjoint operator \hat{A} on \mathcal{H}_x defined at the end of the previous section, we show that the wave operators for the pair of dynamics $\exp(-it\hat{A})$ and $\exp(-it\hat{A}_0)$ exist and are intimately related to the eigenfunction transform \mathcal{F} . As a corollary, this yields isometry of \mathcal{F} . The following lemma is the key to these results.

Lemma 3.1. *For all $\phi \in C_0^\infty((0, \infty))$ we have*

$$\lim_{t \rightarrow \infty} \|(\mathcal{F} - \mathcal{F}_0) \exp(-itM)\phi\| = 0, \quad (3.1)$$

$$\lim_{t \rightarrow -\infty} \|(\mathcal{F} - \mathcal{F}_0 a(\cdot)) \exp(-itM)\phi\| = 0, \quad (3.2)$$

where M is given by (1.13) and $a(\cdot)$ is the operator of multiplication by $a(p)$ (2.13). For all $\phi \in C_0^\infty((-\infty, 0))$ we have

$$\lim_{t \rightarrow \infty} \|(\mathcal{F} - \mathcal{F}_0 a(\cdot)) \exp(-itM)\phi\| = 0, \quad (3.3)$$

$$\lim_{t \rightarrow -\infty} \|(\mathcal{F} - \mathcal{F}_0) \exp(-itM)\phi\| = 0. \quad (3.4)$$

Proof. To prove (3.1), we fix $\phi(p)$ with $\text{supp}(\phi) \subset [r, R]$, $0 < r < R$. Then we have

$$\|(\mathcal{F} - \mathcal{F}_0) \exp(-itM)\phi\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left| \sum_{n=1}^N R_n(x) \int_r^R dp \frac{e^{ixp-2it \cosh p}}{e^p - e^{-r_n}} \phi(p) \right|^2. \quad (3.5)$$

Now when we change variables $p \rightarrow y = \cosh p$, we see that the p -integrals yield bounded functions $b_n(t, x)$ that converge to 0 as $t \rightarrow \infty$ by virtue of the Riemann–Lebesgue lemma. By dominated convergence, this entails that the x -integral over a bounded region converges to 0 for $t \rightarrow \infty$.

To exploit this, we write

$$\int_{-\infty}^{\infty} dx = \int_{-\infty}^{-1} dx + \int_{-1}^1 dx + \int_1^{\infty} dx. \quad (3.6)$$

As we have just shown, the middle integral tends to 0 for $t \rightarrow \infty$. To handle the right integral, we recall from Part I that $R_n(x)$ has exponential decay for $x \rightarrow \infty$, cf. I(2.41), (2.42). This decay supplies the domination we need, in combination with the pointwise convergence to 0 of the x -integrand, to deduce it tends to 0 for $t \rightarrow \infty$, too.

To handle the left integral, we use a stationary phase argument. Specifically, we write

$$\exp(ixp - 2it \cosh p) = (ix - 2it \sinh p)^{-1} \partial_p \exp(ixp - 2it \cosh p), \quad (3.7)$$

and integrate by parts to get

$$\frac{1}{2\pi} \int_{-\infty}^{-1} dx \left| \sum_{n=1}^N R_n(x) \int_r^R dp e^{ixp-2it \cosh p} \partial_p \left(\frac{1}{(x - 2t \sinh p)} \frac{\phi(p)}{(e^p - e^{-r_n})} \right) \right|^2. \quad (3.8)$$

Now we use the estimate

$$|x - 2t \sinh p|^2 \geq x^2 + 4t^2 \sinh^2 r, \quad x \in (-\infty, -1], \quad p \in [r, R], \quad t > 0, \quad (3.9)$$

and boundedness of $R_n(x)$ on \mathbb{R} to obtain an upper bound of the form

$$C \int_{-\infty}^{-1} dx \frac{1}{x^2 + ct^2}, \quad C, c > 0. \quad (3.10)$$

The integrand is bounded above by the $L^1((-\infty, -1])$ -function $1/x^2$ and tends to 0 as $t \rightarrow \infty$. Hence (3.10) tends to 0 as well, so that (3.8) does, too. Therefore, we have now proved (3.1).

In order to prove (3.4), we observe that the estimate (3.9) is also valid for $p \in [-R, -r]$ and $t < 0$. Thus we can choose $\phi(p)$ with $\text{supp}(\phi) \subset [-R, -r]$ and proceed in the same way as for (3.1).

Next, we prove (3.2). To this end we recall (2.49) and (2.50), which we rewrite as

$$\mathcal{W}(x, p) - e^{ixp} a(p) = \sum_{n=1}^N \xi_n(x) / (e^p - e^{-r_n}), \quad (3.11)$$

with $\xi_n(x)$ admitting the two representations

$$\xi_n(x) = (C^{-1}\zeta)_n - R_n(x), \quad (3.12)$$

$$\xi_n(x) = (C^{-1}D(x)R(x))_n. \quad (3.13)$$

Now for $\phi(p)$ with $\text{supp}(\phi) \subset [r, R]$ we have

$$\begin{aligned} & \|(\mathcal{F} - \mathcal{F}_0 a(\cdot)) \exp(-itM)\phi\|^2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left| \sum_{n=1}^N \xi_n(x) \int_r^R dp \frac{e^{ixp - 2it \cosh p}}{e^p - e^{-r_n}} \phi(p) \right|^2. \end{aligned} \quad (3.14)$$

From (3.12) we see that $\xi_n(x)$ is bounded on \mathbb{R} . Using once more the splitting (3.6), it follows that the middle integral tends to 0 as $t \rightarrow -\infty$. The right integral can be handled by the same stationary phase argument as before, noting that the estimate (3.9) is also valid for $x \in [1, \infty)$ and $t < 0$.

It remains to show that the left integral tends to 0 as $t \rightarrow -\infty$. To this end we use the second representation (3.13) of $\xi_n(x)$. Indeed, it entails that $\xi_n(x)$ has exponential decay as $x \rightarrow -\infty$. (Recall the definitions (2.6), (2.7).) Thus we can once again combine the Riemann–Lebesgue lemma and the dominated convergence theorem to deduce convergence to 0 for $t \rightarrow -\infty$.

Finally, to prove (3.3) we can proceed in the same way as for (3.2), noting that (3.9) also holds for $x \in [1, \infty)$ and $p \in [-R, -r]$. \blacksquare

We are now in the position to obtain the principal result of this section.

Theorem 3.2. *The eigenfunction transform \mathcal{F} (1.18) is isometric. The strong limits of the operator family $\exp(it\hat{A}) \exp(-it\hat{A}_0) \mathcal{F}_0$ for $t \rightarrow \pm\infty$ exist and are given by*

$$U_{\pm} = \mathcal{F} \mathcal{A}_{\pm}(\cdot), \quad (3.15)$$

where

$$\mathcal{A}_+(p) \equiv \begin{cases} 1, & p > 0, \\ 1/a(p), & p < 0, \end{cases} \quad (3.16)$$

$$\mathcal{A}_-(p) \equiv \begin{cases} 1/a(p), & p > 0, \\ 1, & p < 0. \end{cases} \quad (3.17)$$

The S -operator

$$S_p \equiv U_+^* U_- \quad (3.18)$$

equals the unitary multiplication operator

$$T(p) \equiv \begin{cases} 1/a(p), & p > 0, \\ a(p), & p < 0. \end{cases} \quad (3.19)$$

Proof. We choose $\psi(p) \in C_0^\infty(\mathbb{R}^*)$, so that

$$\psi = \phi_+ + \phi_-, \quad \phi_+ \in C_0^\infty((0, \infty)), \quad \phi_- \in C_0^\infty((-\infty, 0)). \quad (3.20)$$

By virtue of (3.1) and (3.3) we have

$$\begin{aligned} \|\mathcal{F}\psi - \exp(it\hat{A}) \exp(-it\hat{A}_0) \mathcal{F}_0(\phi_+ + a(\cdot)\phi_-)\| &\leq \|(\mathcal{F} - \exp(it\hat{A}) \exp(-it\hat{A}_0) \mathcal{F}_0)\phi_+\| \\ &+ \|(\mathcal{F} - \exp(it\hat{A}) \exp(-it\hat{A}_0) \mathcal{F}_0 a(\cdot))\phi_-\| \leq \|(\mathcal{F} - \mathcal{F}_0) \exp(-itM)\phi_+\| \\ &+ \|(\mathcal{F} - \mathcal{F}_0 a(\cdot)) \exp(-itM)\phi_-\| \rightarrow 0, \quad t \rightarrow \infty. \end{aligned} \quad (3.21)$$

From this we deduce

$$\begin{aligned} \|\mathcal{F}\psi\| &= \lim_{t \rightarrow \infty} \|\exp(it\hat{A}) \exp(-it\hat{A}_0) \mathcal{F}_0(\phi_+ + a(\cdot)\phi_-)\| \\ &= \|\phi_+ + a(\cdot)\phi_-\| = \|\psi\|. \end{aligned} \quad (3.22)$$

Since $C_0^\infty(\mathbb{R}^*)$ is dense in \mathcal{H}_p , it follows that \mathcal{F} is an isometry. From (3.21) and its analog for $t \rightarrow -\infty$ we also obtain the second assertion of the theorem. The last assertion is then clear from (3.15) and isometry of \mathcal{F} . \blacksquare

4 Bound states and spectral resolution

We begin this section by focusing on the A -eigenfunctions $\mathcal{W}(x, r_k + 2\pi il)$, with $x \in \mathbb{R}$, $k = 1, \dots, N$ and $l \in \mathbb{Z}$, assuming only (2.1)–(2.4). Using (2.11) and (2.5)–(2.8) we see they can be written

$$\mathcal{W}(x, r_k + 2\pi il) = \exp(ir_k x - 2\pi l x) d(r_k, \mu_k; x) R_k(x), \quad l \in \mathbb{Z}. \quad (4.1)$$

Now from (2.8) we readily obtain

$$\lim_{\operatorname{Re} x \rightarrow \infty} d(r_k, \mu_k; x) R_k(x) = 1, \quad k = 1, \dots, N. \quad (4.2)$$

For square-integrability of (4.1) near ∞ we should therefore take $l \in \mathbb{N}$ when $\operatorname{Im} r_k \in (0, \pi)$ and $l \in \mathbb{N}^*$ when $\operatorname{Im} r_k \in (-\pi, 0)$.

Consider now square-integrability near $-\infty$. Since $R_k(x)$ tends to $(C^{-1}\zeta)_k \in \mathbb{C}^*$ for $x \rightarrow -\infty$ (cf. (2.12) and I Lemma 2.1), we see from the definition (2.7) of d that we need $-l \in \mathbb{N}$. For $\text{Im } r_k \in (-\pi, 0)$, therefore, we cannot simultaneously have square-integrability of (4.1) near ∞ and near $-\infty$. In contrast, for $\text{Im } r_k \in (0, \pi)$, the choice $l = 0$ ensures square-integrability near $\pm\infty$.

Thus far, we have not imposed restrictions on (r, μ) beyond our standing assumptions (2.1)–(2.4). But to ensure square-integrability over \mathbb{R} of $\mathcal{W}(x, r_k)$ for $\text{Im } r_k \in (0, \pi)$, we should obviously require absence of poles for real x (the assumption made in Lemma 2.1). Doing so, we obtain A -eigenfunctions whose restrictions to the real axis are in \mathcal{H}_x . We can only prove pairwise orthogonality of these functions, however, when we make the same assumptions as in Lemma 2.2.

Lemma 4.1. *With the assumption of Lemma 2.1 in force, the functions*

$$\psi_n(x) \equiv \mathcal{W}(x, r_n) = \mu_n(x) \exp(-ir_n x) R_n(x), \quad n = 1, \dots, N_+, \quad (4.3)$$

satisfy

$$\psi_n(x) \sim \exp(ir_n x), \quad \text{Re } x \rightarrow \infty, \quad (4.4)$$

$$\psi_n(x) \sim c_n(C^{-1}\zeta)_n \exp(-ir_n x), \quad \text{Re } x \rightarrow -\infty, \quad (4.5)$$

uniformly for $\text{Im } x$ in compacts, and their restrictions to \mathbb{R} belong to \mathcal{H}_x . Now suppose that the requirements of Lemma 2.2 are met. Then the functions $\psi_1(x), \dots, \psi_{N_+}(x)$ are pairwise orthogonal in \mathcal{H}_x .

Proof. We have already shown square-integrability and the asymptotics (4.4)–(4.5). To prove pairwise orthogonality, we invoke the eigenvalue equations

$$(AW)(x, r_n) = 2 \cosh(r_n) \mathcal{W}(x, r_n), \quad n = 1, \dots, N_+. \quad (4.6)$$

From (2.1), (2.2) and (2.26), we see that the eigenvalues are real and distinct. Letting $n \neq m$, we now use (4.6) to write

$$\begin{aligned} & 2[\cosh r_m - \cosh r_n](\psi_n, \psi_m) \\ &= \int_{-\infty}^{\infty} dx (\psi_n^*(x) [\psi_m(x-i) + V_a(x)\psi_m(x+i) + V_b(x)\psi_m(x)] \\ & \quad - [\psi_n^*(x+i) + V_a^*(x)\psi_n^*(x-i) + V_b^*(x)\psi_n^*(x)] \psi_m(x)). \end{aligned} \quad (4.7)$$

Using (1.7), we can write the rhs as

$$\int_{-\infty}^{\infty} dx (J_{nm}(x) - J_{nm}(x+i)), \quad (4.8)$$

with

$$J_{nm}(x) \equiv \psi_n^*(x) \psi_m(x-i) - V_a(x-i) \psi_n^*(x-i) \psi_m(x). \quad (4.9)$$

We now recall (2.29). It entails we may write

$$\psi_n(x) = \mu_n \exp(-ir_n x) E_n(x) / \tau(x), \quad n = 1, \dots, N_+, \quad (4.10)$$

with $E_n(x)$ entire. Using also (2.41) and (2.31), we deduce that $J_{nm}(x)$ can be rewritten as

$$J_{nm}(x) = \overline{\mu_n} \mu_m \left(\exp(-ir_n x - ir_m(x-i)) \frac{E_n^*(x) E_m(x-i)}{\tau(x-i)^2} - \exp(-ir_n(x-i) - ir_m x) \frac{E_n^*(x-i) E_m(x)}{\tau(x-i)^2} \right). \quad (4.11)$$

This representation shows that $J_{nm}(x)$ has no poles on and inside the contour Γ defined below (2.42) in the proof of Lemma 2.2. Thus we can use the same reasoning as in that proof to infer that (4.8) vanishes. (The vanishing of the boundary terms is here a simple consequence of the asymptotics (4.4), (4.5) and (1.2).) Hence pairwise orthogonality follows from vanishing of the lhs of (4.7). \blacksquare

The lemma just proved together with our next lemma are the key to clarifying the character of the subspace $\text{Ran}(\mathcal{F})^\perp$, when the assumptions of Lemma 2.2 are satisfied. From Theorem 3.2 we already know that in that case \mathcal{F} is an isometry, so that we have

$$\mathcal{F}^* \mathcal{F} = \mathbf{1}, \quad \mathcal{F} \mathcal{F}^* = \mathbf{1} - P, \quad (4.12)$$

with P the projection on $\text{Ran}(\mathcal{F})^\perp$. In the next lemma we obtain a formula for $(\mathcal{F}^* f_1, \mathcal{F}^* f_2)$ with $f_1, f_2 \in C_0^\infty(\mathbb{R})$, from which this projection can be explicitly obtained. To prove the pertinent formula, however, we need only impose our second requirement (2.26), together with absence of poles for real x , cf. Lemma 2.1.

Lemma 4.2. *Assume that the spectral data satisfy (2.26), and assume $R(x)$ has no real poles. Then we have for all $f_1, f_2 \in C_0^\infty(\mathbb{R})$*

$$(\mathcal{F}^* f_1, \mathcal{F}^* f_2) = (f_1, f_2) - \sum_{n=1}^{N_+} \nu_n(f_1, \psi_n)(\psi_n, f_2), \quad (4.13)$$

where ψ_n is defined by (4.3).

Proof. Our starting point is the formula

$$(\mathcal{F}^* f_1, \mathcal{F}^* f_2) = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} dy \overline{f_1(y)} \int_{-\infty}^{\infty} dx f_2(x) \mathcal{I}_R(x, y), \quad (4.14)$$

where

$$\mathcal{I}_R(x, y) \equiv \int_{-R}^R dp \mathcal{W}^*(x, p) \mathcal{W}(y, p), \quad x, y \in \mathbb{R}. \quad (4.15)$$

We are going to exploit that $\mathcal{W}(x, p)$ is the product of the plane wave $\exp(ixp)$ and a function of p that is meromorphic and $2\pi i$ -periodic, cf. (2.11).

For this purpose we define the rectangular contour C connecting $-R$, R , $R + 2\pi i$ and $-R + 2\pi i$ in the p -plane, and put

$$\mathcal{I}_C(x, y) \equiv \oint_C dp \mathcal{W}^*(x, p) \mathcal{W}(y, p), \quad x, y \in \mathbb{R}. \quad (4.16)$$

(Of course, the $*$ refers here to the p -dependence, cf. (1.8).) Due to our requirements on r_1, \dots, r_N , the only singularities of the integrand inside C consist of simple poles at the $2N$ distinct points

$$p = r_n, 2\pi i - r_n, \quad n = 1, \dots, N_+, \quad (4.17)$$

$$p = -r_n, 2\pi i + r_n, \quad n = N_+ + 1, \dots, N, \quad (4.18)$$

on the imaginary axis. Hence Cauchy's theorem yields

$$\mathcal{I}_C(x, y) = 2\pi i \mathcal{R}(x, y), \quad (4.19)$$

where $\mathcal{R}(x, y)$ denotes the sum of the residues.

On the other hand, we can also write

$$\mathcal{I}_C(x, y) = \mathcal{I}_R(x, y) - e^{2\pi(x-y)} \mathcal{I}_R(x, y) + \mathcal{B}_R(x, y), \quad (4.20)$$

with

$$\mathcal{B}_R(x, y) \equiv \left(\int_R^{R+2\pi i} + \int_{-R+2\pi i}^{-R} \right) dp \mathcal{W}^*(x, p) \mathcal{W}(y, p). \quad (4.21)$$

Hence we have

$$\mathcal{I}_R(x, y) = \left(1 - e^{2\pi(x-y)} \right)^{-1} [2\pi i \mathcal{R}(x, y) - \mathcal{B}_R(x, y)]. \quad (4.22)$$

We proceed to calculate the contribution of the residue sum to the inner product

$$\begin{aligned} (\mathcal{F}^* f_1, \mathcal{F}^* f_2) &= \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} dy \overline{f_1(y)} \int_{-\infty}^{\infty} dx f_2(x) \\ &\quad \times \left(1 - e^{2\pi(x-y)} \right)^{-1} [2\pi i \mathcal{R}(x, y) - \mathcal{B}_R(x, y)], \end{aligned} \quad (4.23)$$

where we combined (4.14) and (4.22). First, we show that the residues at the poles (4.18) cancel pairwise. Indeed, fixing $l \in \{N_+ + 1, \dots, N\}$, the residue sum at $p = -r_l, 2\pi i + r_l$ of the pertinent function

$$\exp[ip(y-x)] \left(1 - \sum_{k=1}^N \frac{R_k^*(x)}{e^p - e^{r_k}} \right) \left(1 - \sum_{m=1}^N \frac{R_m(y)}{e^p - e^{-r_m}} \right) \quad (4.24)$$

equals

$$\begin{aligned} &\exp[-ir_l(y-x)] \left(1 - \sum_k \frac{R_k^*(x)}{e^{-r_l} - e^{r_k}} \right) \left(-\frac{R_l(y)}{e^{-r_l}} \right) \\ &+ \exp[(ir_l - 2\pi)(y-x)] \left(-\frac{R_l^*(x)}{e^{r_l}} \right) \left(1 - \sum_m \frac{R_m(y)}{e^{r_l} - e^{-r_m}} \right). \end{aligned} \quad (4.25)$$

Recalling the system (2.8), we see that this equals

$$\begin{aligned} &-\exp[-ir_l(y-x)] d^*(r_l, \mu_l; x) R_l^*(x) e^{r_l} R_l(y) \\ &-\exp[(ir_l - 2\pi)(y-x)] e^{-r_l} R_l^*(x) d(r_l, \mu_l; y) R_l(y). \end{aligned} \quad (4.26)$$

From the definition (2.7) of d we see that this is proportional to

$$\begin{aligned} & \exp[-ir_l(y-x)]\overline{\mu_l} \exp[-2i(r_l+i\pi)x]e^{r_l} \\ & + \exp[(ir_l-2\pi)(y-x)]e^{-r_l}\mu_l \exp[-2i(r_l+i\pi)y] \\ & = \exp[-ir_l(y+x)]e^{2\pi x}(\overline{\mu_l}e^{r_l} + \mu_le^{-r_l}), \end{aligned} \quad (4.27)$$

which vanishes due to (2.26).

We are therefore left with the residues of (4.24) at the points (4.17). The residue sum at $p = r_n, 2\pi i - r_n$ for $n \in \{1, \dots, N_+\}$ equals (using (2.8), (2.26) and (4.3))

$$\begin{aligned} & \exp[ir_n(y-x)]\left(-\frac{R_n^*(x)}{e^{r_n}}\right)\mu_n e^{-2ir_n y}R_n(y) \\ & + \exp[(-ir_n-2\pi)(y-x)]\overline{\mu_n}e^{-2ir_n x}R_n^*(x)\left(-\frac{R_n(y)}{e^{-r_n}}\right) \\ & = -\exp[-ir_n(y+x)]R_n^*(x)R_n(y)\left(\mu_n e^{-r_n} + e^{2\pi(x-y)}\overline{\mu_n}e^{r_n}\right) \\ & = i\nu_n^{-1}\left(1 - e^{2\pi(x-y)}\right)e^{-ir_n x}R_n^*(x)e^{-ir_n y}R_n(y) \\ & = i\nu_n\left(1 - e^{2\pi(x-y)}\right)\overline{\psi_n(x)}\psi_n(y). \end{aligned} \quad (4.28)$$

Substituting this in (4.23) and comparing to (4.13), we deduce that it remains to prove

$$\frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} dy \overline{f_1(y)} \int_{-\infty}^{\infty} dx f_2(x) \frac{\mathcal{B}_R(x, y)}{e^{2\pi(x-y)} - 1} = (f_1, f_2). \quad (4.29)$$

To this end we first rewrite $\mathcal{B}_R(x, y)$ (4.21) as

$$\begin{aligned} \mathcal{B}_R(x, y) &= \int_{R-i\pi}^{R+i\pi} dp [\mathcal{W}^*(x, p+i\pi)\mathcal{W}(y, p+i\pi) - \mathcal{W}^*(x, -p+i\pi)\mathcal{W}(y, -p+i\pi)] \\ &= e^{\pi(x-y)} \int_{R-i\pi}^{R+i\pi} dp \left[e^{ip(y-x)}A(p, x, y) - e^{ip(x-y)}A(-p, x, y) \right], \end{aligned} \quad (4.30)$$

where the auxiliary function A is given by

$$A(p, x, y) \equiv \left(1 + \sum_n \frac{R_n^*(x)}{e^p + e^{r_n}}\right) \left(1 + \sum_m \frac{R_m(y)}{e^p + e^{-r_m}}\right). \quad (4.31)$$

We now claim that the identity

$$A(p, x, x) = A(-p, x, x) \quad (4.32)$$

holds true.

To prove (4.32), we observe that the functions $A(\pm p, x, x)$ are $2\pi i$ -periodic in p and bounded for $|\operatorname{Re} p| \rightarrow \infty$, and that they have simple poles in the period strip $\operatorname{Im} p \in [0, 2\pi]$ at $p = i\pi \pm r_n$, $n = 1, \dots, N$. Using (2.8) and (2.7) in the same way as above (cf. (4.24)–(4.27)), we readily verify that the functions have equal residues. By Liouville's theorem, it now follows that $A(p, x, x) - A(-p, x, x)$ does not depend on p .

To show that this difference vanishes, we need only compare the $\operatorname{Re} p \rightarrow \infty$ limits of $A(\pm p, x, x)$. Obviously, we have

$$\lim_{\operatorname{Re} p \rightarrow \infty} A(p, x, x) = 1, \quad \lim_{\operatorname{Re} p \rightarrow \infty} A(-p, x, x) = \lambda^*(x)\lambda(x), \quad (4.33)$$

where

$$\lambda(x) \equiv 1 + \sum_{m=1}^N e^{r_m} R_m(x). \quad (4.34)$$

Now from I(D.17) we see that (2.26) entails

$$\lambda^*(x)\lambda(x) = 1. \quad (4.35)$$

Hence our claim (4.32) follows.

Next, we introduce functions B and C by setting

$$A(p, x, y) = 1 + e^{-p} B(p, x, y), \quad (4.36)$$

$$A(-p, x, y) = \lambda^*(x)\lambda(y) + e^{-p} C(p, x, y). \quad (4.37)$$

Due to (4.32), these functions are related by

$$B(p, x, x) = C(p, x, x). \quad (4.38)$$

We now rewrite (4.30) as

$$\mathcal{B}_R(x, y) = e^{\pi(x-y)} \left(\mathcal{B}_R^{(d)}(x, y) + \mathcal{B}_R^{(r)}(x, y) \right), \quad (4.39)$$

with

$$\mathcal{B}_R^{(d)}(x, y) \equiv \int_{R-i\pi}^{R+i\pi} dp \left[e^{ip(y-x)} - e^{ip(x-y)} \lambda^*(x)\lambda(y) \right], \quad (4.40)$$

$$\mathcal{B}_R^{(r)}(x, y) \equiv \int_{R-i\pi}^{R+i\pi} dp e^{-p} \left[e^{ip(y-x)} B(p, x, y) - e^{ip(x-y)} C(p, x, y) \right]. \quad (4.41)$$

The point is that we can get rid of the remainder term $\mathcal{B}_R^{(r)}(x, y)$ by using (4.38).

Specifically, (4.38) entails we may write

$$\mathcal{B}_R^{(r)}(x, y) = \int_{R-i\pi}^{R+i\pi} dp e^{-p} \int_x^y ds \partial_s \left(e^{ip(s-x)} B(p, x, s) - e^{ip(x-s)} C(p, x, s) \right). \quad (4.42)$$

Now from the definitions (4.36), (4.37) of B and C we deduce that for all s -values between x and y we have

$$\left| \partial_s \left(e^{ip(s-x)} B(p, x, s) - e^{ip(x-s)} C(p, x, s) \right) \right| \leq |p| D(x, y), \quad (4.43)$$

$$\operatorname{Re} p \geq R, \quad |\operatorname{Im} p| \leq \pi,$$

where $D(x, y)$ is a positive function that is bounded for x, y varying over \mathbb{R} -compacts. Thus we obtain

$$\left| \mathcal{B}_R^{(r)}(x, y) \right| \leq 2\pi e^{-R} (R^2 + \pi^2)^{1/2} |y - x| D(x, y). \quad (4.44)$$

Hence the contribution of $\mathcal{B}_R^{(r)}(x, y)$ to the lhs of (4.29) vanishes.

We are now reduced to showing

$$\lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} dy \overline{f_1(y)} \int_{-\infty}^{\infty} dx f_2(x) \frac{\mathcal{B}_R^{(d)}(x, y)}{\sinh(\pi(x - y))} = 4\pi(f_1, f_2). \quad (4.45)$$

Calculating the integral (4.40), we can write the result as

$$\mathcal{B}_R^{(d)}(x, y) = 2 \sinh(\pi(x - y)) [C_c(R; x, y) + C_s(R; x, y)], \quad (4.46)$$

where

$$C_c(R; x, y) \equiv \left(\frac{1 - \lambda^*(x)\lambda(y)}{i(y - x)} \right) \cos(y - x)R, \quad (4.47)$$

$$C_s(R; x, y) \equiv (1 + \lambda^*(x)\lambda(y)) \frac{\sin(y - x)R}{y - x}. \quad (4.48)$$

By virtue of (4.35) and the Riemann–Lebesgue lemma, the contribution of $C_c(R; x, y)$ to (4.45) vanishes. Using the tempered distribution limit

$$\lim_{c \rightarrow \infty} \frac{\sin cx}{x} = \pi \delta(x) \quad (4.49)$$

and (4.35), the remaining term $C_s(R; x, y)$ yields (4.45). \blacksquare

Clearly, we can rewrite (4.13) as the operator identity

$$\mathcal{F}\mathcal{F}^* = \mathbf{1} - \sum_{n=1}^{N_+} \nu_n \psi_n \otimes \overline{\psi_n}, \quad (4.50)$$

where the rank-one operator $\psi \otimes \chi$, with $\psi, \chi \in \mathcal{H}_x$, is defined by

$$(\psi \otimes \chi)f = (\overline{\chi}, f)\psi, \quad f \in \mathcal{H}_x. \quad (4.51)$$

For the remainder of this section, we assume that the requirements of Lemma 2.2 are met. Then \mathcal{F} is an isometry (as proved in Theorem 3.2), and the functions $\psi_1, \dots, \psi_{N_+}$ (4.3) are pairwise orthogonal in \mathcal{H}_x (as proved in Lemma 4.1). In view of (4.50), the projection P on $\text{Ran}(\mathcal{F})^\perp$ can be written

$$P = \sum_{n=1}^{N_+} \nu_n \psi_n \otimes \overline{\psi_n}. \quad (4.52)$$

In particular, this entails the norm formula

$$(\psi_n, \psi_n) = \nu_n^{-1}, \quad n = 1, \dots, N_+. \quad (4.53)$$

We are now in the position to turn the provisional definition of the self-adjoint Hilbert space operator \hat{A} (see the end of Section 2) into a final one: We define \hat{A} on $\text{Ran}(\mathcal{F})^\perp$ by (linear extension of)

$$\hat{A}\psi_n \equiv 2 \cosh(r_n)\psi_n, \quad n = 1, \dots, N_+. \quad (4.54)$$

Theorem 4.3. *The operator \hat{A} is essentially self-adjoint on the dense subspace*

$$\mathcal{C} \equiv \mathcal{P} \oplus \text{Span}(\psi_1, \dots, \psi_{N_+}), \quad (4.55)$$

and its action on \mathcal{C} coincides with that of the A Δ O A . The operator \hat{A} has absolutely continuous spectrum $[2, \infty)$ with multiplicity two, and point spectrum $\{2 \cosh(r_1), \dots, 2 \cosh(r_{N_+})\}$ with multiplicity one.

Proof. The first assertion follows by combining Lemmas 2.1 and 2.2 with (4.3), (4.6) and (4.54). Denoting the domain of \hat{A} by \mathcal{D} , the restriction of \hat{A} to $\mathcal{D} \cap \text{Ran}(\mathcal{F})$ is unitarily equivalent to multiplication by $2 \cosh(p)$ on \mathcal{H}_p , cf. (1.22). Together with the distinctness of the numbers $r_1, \dots, r_{N_+} \in i(0, \pi)$, this entails the second assertion. ■

We conclude this section with some further observations concerning three special cases. Taking first $N_- = 0$, we recall that the assumptions

$$0 < -ir_1 < \dots < -ir_N < \pi, \quad \nu_n \in (0, \infty), \quad n = 1, \dots, N, \quad (4.56)$$

are sufficient for all of our results to hold true, cf. Lemma A.1. Now in this special case the A Δ O A satisfies

$$A = S_+^2 - 2, \quad (4.57)$$

where S_+ is the A Δ O

$$S_+ \equiv \exp(-i\partial_x/2) + V(x) \exp(i\partial_x/2), \quad (4.58)$$

$$V(x) \equiv \lambda(x)/\lambda(x + i/2), \quad (4.59)$$

with $\lambda(x)$ given by (4.34); moreover,

$$(S_+ \mathcal{W})(x, p) = \left(e^{p/2} + e^{-p/2} \right) \mathcal{W}(x, p). \quad (4.60)$$

(These assertions follow from I Theorem 3.3.) Using the above results, we can associate a self-adjoint operator \hat{S}_+ to S_+ by setting

$$\hat{S}_+ \mathcal{F} f \equiv \mathcal{F} M_+ f, \quad (M_+ f)(p) \equiv 2 \cosh(p/2) f(p), \quad f \in \mathcal{D}(M_+), \quad (4.61)$$

$$\hat{S}_+ \psi_n \equiv 2 \cosh(r_n/2) \psi_n, \quad n = 1, \dots, N_+. \quad (4.62)$$

It follows just as for A that the dense subspace \mathcal{C} (4.55) is a core for \hat{S}_+ on which the \hat{S}_+ -action coincides with that of S_+ .

Secondly, we consider the special case $N = 2M$, $N_+ = N_- = M$, together with spectral data

$$\begin{aligned} 0 < -ir_1 < \dots < -ir_M < \pi/2, \quad \nu_j \in (0, \infty), \\ r_{N-j+1} = r_j - i\pi, \quad \nu_{N-j+1} = \nu_j, \end{aligned} \quad (4.63)$$

where $j = 1, \dots, M$. Again, this suffices for all of the above Hilbert space results to be valid, cf. Lemma A.2. In terms of the particle variables defined in Appendix A, this choice of spectral data amounts to

$$0 < q_M^+ < \dots < q_1^+, \quad q_j^- = -q_{M-j+1}^+, \quad \theta_j^- = \theta_{M-j+1}^+, \quad j = 1, \dots, M. \quad (4.64)$$

Its distinguishing feature consists in the potential $V_b(x)$ being identically zero. Moreover, after taking $M \rightarrow N$ and performing a scaling $x, p \rightarrow 2x, p/2$, the class of AΔOs A , together with their reflectionless eigenfunctions and associated self-adjoint operators \hat{A} , amounts to the class of AΔOs S_+ , together with their reflectionless eigenfunctions and associated self-adjoint operators \hat{S}_+ . Once more, this follows from I Theorem 3.3. (The ordering we used there is different, but this is inconsequential. Indeed, all of the pertinent quantities are permutation invariant.)

Thirdly, we consider the special case $N_+ = 0$. From Lemma A.1 we then infer that the assumptions

$$-\pi < -ir_N < \cdots < -ir_1 < 0, \quad \nu_n \in (0, \infty), \quad n = 1, \dots, N, \quad (4.65)$$

suffice for the validity of our Hilbert space results. In this case we have the AΔO identity

$$A = S_-^2 + 2, \quad (4.66)$$

where

$$S_- \equiv \exp(-i\partial_x/2) - V(x) \exp(i\partial_x/2), \quad (4.67)$$

and $V(x)$ is again given by (4.59); moreover,

$$(S_- \mathcal{W})(x, p) = \left(e^{p/2} - e^{-p/2} \right) \mathcal{W}(x, p). \quad (4.68)$$

(These assertions are also a consequence of I Theorem 3.3.) Since \mathcal{F} is unitary when N_+ vanishes, we can define a self-adjoint operator \hat{S}_- by

$$\hat{S}_- \equiv \mathcal{F} M_- \mathcal{F}^*, \quad (M_- f)(p) \equiv 2 \sinh(p/2) f(p), \quad f \in \mathcal{D}(M_-). \quad (4.69)$$

As before, the subspace $\mathcal{C} = \mathcal{P}$ is a core for \hat{S}_- , on which the \hat{S}_- -action coincides with that of S_- .

We would like to point out that the absence of bound states for the case $N_+ = 0$ is a quite remarkable feature. Indeed, for reflectionless self-adjoint Schrödinger and Jacobi operators, absence of bound states implies that the potentials are trivial (constant), whereas here one obtains an infinite-dimensional family of nontrivial potential pairs V_a, V_b . When one takes the time dependence introduced in Part II into account, this family of reflectionless self-adjoint operators without bound states yields the left-moving soliton solutions to the analytic version of the Toda lattice studied in Part II.

We can also use the $N_+ = 0$ special case to illustrate the ambiguity issue discussed in the introduction, cf. in particular the paragraph below (1.18). Let us begin by noting that all of the wave functions $\mathcal{W}(x, p)$ studied in this series of papers satisfy

$$(D\mathcal{W})(x, p) = 2 \cosh(2\pi x) \mathcal{W}(x, p), \quad (4.70)$$

where the dual AΔO D is given by

$$D \equiv \exp(2\pi i \partial_p) + \exp(-2\pi i \partial_p). \quad (4.71)$$

Indeed, this is plain from $\mathcal{W}(x, p)$ being the product of the factor $\exp(ixp)$ and a factor that is a rational function of $\exp(p)$, cf. (2.11). For $N_+ = 0$ the operator \mathcal{F} is unitary, so we can define a self-adjoint operator \hat{D} on \mathcal{H}_p by setting

$$\hat{D} \equiv \mathcal{F}^* M_x \mathcal{F}, \quad (M_x f)(x) \equiv 2 \cosh(2\pi x) f(x), \quad f \in \mathcal{D}(M_x). \quad (4.72)$$

The action of \hat{D} on the core $\mathcal{F}^*(C_0^\infty(\mathbb{R}))$ now coincides with the action of the A Δ O D . (This follows in the same way as the analogous assertion in Lemma 2.2.)

The upshot is that we have associated to the free A Δ O D an infinite-dimensional family of distinct self-adjoint reflectionless operators \hat{D} without bound states. (Indeed, the function $\lambda(x)$ (4.34) plays the same role for \hat{D} as the function $a(p)$ (2.13) plays for \hat{A} .) Interchanging x and p and performing a scaling by 2π , we see that we obtain a similar family associated with the free A Δ O A_0 (1.11), as announced.

Finally, we would like to use D with $N_+ > 0$ to illustrate that Hilbert space operators associated to A Δ Os may look symmetric at first sight, even when they are not symmetric. Indeed, the symmetry property is far more elusive than may be apparent from the above results. (For instance, symmetry is probably generically violated when the requirements of Lemma 2.2 are not met, cf. also our results in Ref. [3].)

For this purpose we observe that whenever the assumption of Lemma 2.1 is satisfied, we may define an operator \hat{D} on the subspace $\mathcal{F}^*(C_0^\infty(\mathbb{R}))$ via

$$\hat{D}\mathcal{F}^*f \equiv \mathcal{F}^*M_x f, \quad f \in C_0^\infty(\mathbb{R}). \quad (4.73)$$

(The point is that we have

$$\text{Ker}(\mathcal{F}^*) \cap C_0^\infty(\mathbb{R}) = \{0\}, \quad (4.74)$$

by the argument proving (2.15); hence \hat{D} is well defined.) Now with the stronger requirements of Lemma 2.2 in effect, \mathcal{F} is isometric, so that $\mathcal{F}^*(C_0^\infty(\mathbb{R}))$ is dense in \mathcal{H}_p . For $N_+ > 0$, however, the densely defined operator \hat{D} is *not* symmetric.

This assertion can be verified in two ways, both of which are illuminating. First, we can proceed as in the proof of Lemma 2.2 to try and show symmetry. Doing so, we are led to investigate the residue sum of the function (4.24) in the strip $\text{Im } p \in [0, 2\pi]$. As we have seen below (4.24), this residue sum vanishes only when $N_+ = 0$, so that \hat{D} is not symmetric for $N_+ > 0$.

The second way in which symmetry violation for $N_+ > 0$ can be established hinges on the finite-dimensionality of $\text{Ran}(\mathcal{F})^\perp$ already detailed above. Specifically, $\text{Ran}(\mathcal{F})^\perp$ is N_+ -dimensional, and this suffices to rule out symmetry for $N_+ > 0$.

To see this, assume \hat{D} is symmetric on $\mathcal{F}^*(C_0^\infty(\mathbb{R}))$. Then it follows as before that \hat{D} is essentially self-adjoint on $\mathcal{F}^*(C_0^\infty(\mathbb{R}))$, and also that we have

$$\exp(it\hat{D})\mathcal{F}^*f = \mathcal{F}^*\exp(itM_x)f, \quad f \in C_0^\infty(\mathbb{R}), \quad t \in \mathbb{C}. \quad (4.75)$$

Choosing t real and taking closures, this entails

$$\exp(it\overline{\hat{D}})\mathcal{F}^* = \mathcal{F}^*\exp(itM_x), \quad (4.76)$$

and so

$$\exp(it\overline{\hat{D}}) = \mathcal{F}^*\exp(itM_x)\mathcal{F}, \quad t \in \mathbb{R}, \quad (4.77)$$

by isometry of \mathcal{F} . From this we deduce that the unitary one-parameter group $\exp(itM_x)$ leaves $\text{Ran}(\mathcal{F})$ invariant. Hence it leaves $\text{Ran}(\mathcal{F})^\perp$ invariant, too. Since the generator M_x has solely continuous spectrum and $\text{Ran}(\mathcal{F})^\perp$ is N_+ -dimensional, we must have $N_+ = 0$, as advertized.

Appendix A. Absence of $\tau(x)$ -zeros for $\operatorname{Im} x \in [-1, 0]$

In this appendix we state and prove two lemmas that have a bearing on the most restrictive (fourth) requirement made in Section 2. We recall that this requirement consists in the restrictions (2.26)–(2.27) on the spectral data $(r, \mu(x))$ given by the first paragraph of Section 2, and in the additional restriction that $\tau(x)$ (2.28) have no zeros for $\operatorname{Im} x \in [-1, 0]$. Our first lemma shows that when N_+ or N_- vanishes, there is no need for the latter restriction.

Lemma A.1. *Assume that the spectral data satisfy*

$$0 < -ir_1 < \cdots < -ir_N < \pi, \quad \nu_n \in (0, \infty), \quad n = 1, \dots, N, \quad (\text{A.1})$$

or

$$-\pi < -ir_N < \cdots < -ir_1 < 0, \quad \nu_n \in (0, \infty), \quad n = 1, \dots, N, \quad (\text{A.2})$$

where ν_n is given by (2.26). Then $\tau(x)$ does not vanish in the strip $\operatorname{Im} x \in [-1, 0]$.

For $N_+N_- > 0$, however, it seems likely that zeros of $\tau(x)$ in the critical strip are not excluded by (2.26)–(2.27). (In particular, for $N_+ = N_- = 1$, we can show that $\tau(x)$ may have zeros at the strip boundaries.) But an extra restriction on r ensures absence of critical zeros.

Lemma A.2. *Assume that the spectral data satisfy $N_+N_- > 0$ and that in addition to (2.26)–(2.27) we have*

$$0 < -ir_1 < \cdots < -ir_{N_+} \leq \frac{\pi}{2}, \quad -\pi < -ir_N < \cdots < -ir_{N_++1} \leq -\frac{\pi}{2}. \quad (\text{A.3})$$

Then $\tau(x)$ has no zeros for $\operatorname{Im} x \in [-1, 0]$.

To prove these lemmas we invoke Section 5 in Part II. To this end, we require from now on (2.26)–(2.27). Then we put

$$\alpha_j^+ \equiv -ir_j, \quad j = 1, \dots, N_+, \quad \alpha_l^- \equiv -ir_{N_++l} + \pi, \quad l = 1, \dots, N_-, \quad (\text{A.4})$$

so that

$$\alpha_n^\delta \in (0, \pi), \quad n = 1, \dots, N_\delta, \quad \delta = +, -. \quad (\text{A.5})$$

Next, we define real numbers

$$\begin{aligned} q_j^+ &\equiv \ln(\cot(\alpha_j^+/2)), & j = 1, \dots, N_+, \\ q_l^- &\equiv -\ln(\cot(\alpha_l^-/2)), & l = 1, \dots, N_-, \end{aligned} \quad (\text{A.6})$$

and positive ‘potentials’

$$\begin{aligned} V_j^+(q) &\equiv \prod_{1 \leq k \leq N_+, k \neq j} |\coth[(q_j^+ - q_k^+)/2]| \\ &\times \prod_{1 \leq l \leq N_-} |\tanh[(q_j^+ - q_l^-)/2]|, & j = 1, \dots, N_+, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} V_l^-(q) &\equiv \prod_{1 \leq m \leq N_-, m \neq l} |\coth[(q_l^- - q_m^-)/2]| \\ &\times \prod_{1 \leq j \leq N_+} |\tanh[(q_l^- - q_j^+)/2]|, & l = 1, \dots, N_-. \end{aligned} \quad (\text{A.8})$$

Finally, we introduce real numbers θ_j^+ , $j = 1, \dots, N_+$, θ_l^- , $l = 1, \dots, N_-$, by writing ν_n as

$$\nu_j = \frac{2V_j^+(q)}{\cosh(q_j^+)} \exp(\theta_j^+), \quad j = 1, \dots, N_+, \quad (\text{A.9})$$

$$\nu_{N_++l} = \frac{2V_l^-(q)}{\cosh(q_l^-)} \exp(\theta_l^-), \quad l = 1, \dots, N_-. \quad (\text{A.10})$$

With this reparametrization of the spectral data in effect, the tau-function $\tau(x)$ depends on points in the phase space

$$\Omega \equiv \left\{ (q_1^+, \dots, q_{N_-}^-, \theta_1^+, \dots, \theta_{N_-}^-) \in \mathbb{R}^{2N} \mid q_{N_+}^+ < \dots < q_1^+, \right. \\ \left. q_{N_-}^- < \dots < q_1^-, q_j^+ \neq q_l^-, j = 1, \dots, N_+, l = 1, \dots, N_- \right\} \quad (\text{A.11})$$

of the $\tilde{\Pi}_{\text{rel}}(\tau = \pi/2)$ system studied in Ref. [7]. The crux is now that it can be rewritten as

$$\tau(x) = |\mathbf{1}_N + L(x + i/2)|, \quad (\text{A.12})$$

where

$$L(x) \equiv \mathcal{L}(q^+, q^-, \theta_1^+ - 2\alpha_1^+ x, \dots, \theta_{N_-}^- - 2\alpha_{N_-}^- x), \quad (\text{A.13})$$

and \mathcal{L} may be viewed as the Lax matrix of this integrable N -particle system. Specifically, we have (cf. II Section 5)

$$\mathcal{L}(q, \theta) \equiv \mathcal{C}(q^+, q^-) \mathcal{D}(q^+, q^-, \theta^+, \theta^-), \quad (\text{A.14})$$

where \mathcal{D} is the diagonal matrix

$$\mathcal{D} \equiv \text{diag} \left(\exp(\theta_1^+) V_1^+(q), \dots, \exp(\theta_{N_+}^+) V_{N_+}^+(q), \right. \\ \left. \exp(\theta_1^-) V_1^-(q), \dots, \exp(\theta_{N_-}^-) V_{N_-}^-(q) \right), \quad (\text{A.15})$$

and \mathcal{C} is the Cauchy matrix

$$\mathcal{C}_{jk} \equiv 1 / \cosh[(q_j^+ - q_k^+)/2], \quad (\text{A.16})$$

$$\mathcal{C}_{N_++l, N_++m} \equiv 1 / \cosh[(q_l^- - q_m^-)/2], \quad (\text{A.17})$$

$$\mathcal{C}_{N_++l, k} \equiv -i / \sinh[(q_l^- - q_k^+)/2], \quad (\text{A.18})$$

$$\mathcal{C}_{j, N_++m} \equiv i / \sinh[(q_j^+ - q_m^-)/2], \quad (\text{A.19})$$

with $j, k = 1, \dots, N_+$, $l, m = 1, \dots, N_-$. We are now prepared to prove the lemmas.

Proofs of Lemmas A.1–A.2. We fix $\xi \in \mathbb{R}$ and note that we may write

$$\tau(\xi - i/2 + i\eta) = |\mathbf{1}_N + L(\xi) U^*(\eta)|, \quad (\text{A.20})$$

where $U(\eta)$ is the diagonal unitary matrix

$$U(\eta) \equiv \text{diag}(\exp(2i\alpha_1^+ \eta), \dots, \exp(2i\alpha_{N_-}^- \eta)), \quad \eta \in \mathbb{R}. \quad (\text{A.21})$$

Putting

$$F(\eta) \equiv |U(\eta) + L(\xi)|, \quad (\text{A.22})$$

we should show that $F(\eta)$ does not vanish for any $\eta \in [-1/2, 1/2]$. For this purpose we first note that

$$\mathcal{D}(\xi) \equiv \text{diag}(\exp(\theta_1^+ - 2\alpha_1^+ \xi)V_1^+, \dots, \exp(\theta_{N_-}^- - 2\alpha_{N_-}^- \xi)V_{N_-}^-) \quad (\text{A.23})$$

is positive. Similarity transforming $L(\xi)$ with the positive diagonal matrix $\mathcal{D}(\xi)^{1/2}$, we obtain the symmetrized Lax matrix

$$L^s(\xi) \equiv \mathcal{D}(\xi)^{1/2} \mathcal{C} \mathcal{D}(\xi)^{1/2}. \quad (\text{A.24})$$

Since $U(\eta)$ is diagonal, we have

$$F(\eta) = |U(\eta) + L^s(\xi)|. \quad (\text{A.25})$$

Consider now the inner product $(f, L^s(\xi)f)$ for a nonzero $f \in \mathbb{C}^N$. When N_+ or N_- vanishes, the matrix \mathcal{C} is positive, so that $(f, L^s(\xi)f) > 0$. With (A.5) in force, we also have the inequalities

$$\delta \text{Im}(f, U(\eta)f) > 0, \quad \delta\eta \in (0, 1/2], \quad f \neq 0, \quad \delta = +, -. \quad (\text{A.26})$$

Therefore, $\text{Im}(f, (U(\eta) + L^s(\xi))f)$ does not vanish for $|\eta| \in (0, 1/2]$ and $f \neq 0$. This entails that $U(\eta) + L^s(\xi)$ is regular for $|\eta| \in (0, 1/2]$. Since $\mathbf{1}_N + L^s(\xi)$ is also regular, $F(\eta)$ (A.25) does not vanish for $\eta \in [-1/2, 1/2]$, and so Lemma A.1 follows.

The restrictions (A.3) in Lemma A.2 amount to

$$\alpha_n^\delta \in (0, \pi/2], \quad n = 1, \dots, N_\delta, \quad \delta = +, -. \quad (\text{A.27})$$

Hence we deduce

$$\text{Re}(f, U(\eta)f) \geq 0, \quad |\eta| \leq 1/2, \quad f \in \mathbb{C}^N. \quad (\text{A.28})$$

Now for $N_+N_- > 0$ it is no longer true that \mathcal{C} is positive. But in view of (A.16)–(A.19) we may write

$$\mathcal{C} = \mathcal{C}_{++} + \mathcal{C}_{--} + i\mathcal{C}^s, \quad (\text{A.29})$$

with $\mathcal{C}_{++} + \mathcal{C}_{--}$ positive and \mathcal{C}^s real and symmetric. From this it readily follows that we have

$$\text{Re}(f, L^s(\xi)f) > 0, \quad f \neq 0. \quad (\text{A.30})$$

Combined with (A.28), this entails that $U(\eta) + L^s(\xi)$ is regular for $|\eta| \leq 1/2$. Hence Lemma A.2 follows. \blacksquare

We would like to add that this proof involves only superficial features of the symmetrized Lax matrix. Presumably, the far more detailed information obtained in Ref. [7] can be used to relax the requirement (A.3).

Acknowledgments

This paper was completed during our stay at the Newton Institute for Mathematical Sciences (September–October 2001, Integrable Systems Programme). We would like to thank the Institute for its hospitality and financial support.

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